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HYPERSURFACE VARIATIONS ARE MAXIMAL, II

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ABSTRACT. We show that certain variations of Hodge structure defined by sufficiently ample hypersurfaces are maximal integral manifolds of Griffiths' horizontal distribution.

1. INTRODUCTION

Griffiths' infinitesimal period relation,

$$\frac{\partial}{\partial z_j} F^p \subset F^{p-1},$$

defines a distribution $\mathcal{G}$ in the holomorphic tangent bundle of the classifying spaces D for polarized Hodge structures [13]. The variations of Hodge structure are precisely the closed integral submanifolds of $\mathcal{G}$ which are stable under the action of a discrete group $\Gamma \subset G$, where G is the isometry group of D. Among such manifolds one may distinguish the "maximal" elements: those which are not contained in any integral manifold of strictly larger dimension. According to the results of [2], the variations defined by hypersurfaces are maximal with a small list of exceptions (plane curves of degree > 4, cubic three- and four-folds, and the quintic three-fold). Here we show that the same holds for hypersurfaces in any ambient space of dimension greater than 3 which, like projective space, is rigid and has zero first Betti number:

(1.1) **Theorem.** *Let Y be a smooth projective variety of dimension $n > 3$ with $H^1(Y, \mathbb{Q}) = 0$ and $H^1(Y, \Theta) = 0$. Let L be a line bundle on Y, $|L|_0$ the variety of smooth divisors of L, and Z an element of $|L|_0$. Set*

$$H^m(Z)_0 = \text{kernel}\{Gysin\colon H^m(Z) \to H^{m+2}(Y)\},$$

where $m = n - 1$ is the dimension of Z. Then the variation of Hodge structures over $|L|_0$ defined by $Z \mapsto H^n(Z)_0$ is maximal as an integral manifold of Griffiths' distribution, provided that L is sufficiently ample. If $n = 3$ the proposition also holds, provided that $H^{3,0}(Y) = 0$.

The hypothesis that Y be rigid is necessary: Consider a complete intersection $Z = X_1 \cap \cdots \cap X_s$, where X_i is a hypersurface of degree d_i in $\mathbb{P}^n$. According

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to [16, Theorem 5.4], the local Torelli theorem holds for the family of all such Z if the canonical bundle is positive, i.e., if $d_1 + \cdots + d_s > n+1$. Moreover, the primitive cohomology of Y, taken in the usual sense of "complement of the hyperplane class", agrees with that defined above for the inclusion of Z in $Y = X_2 \cap \cdots \cap X_s$ [16, Theorem 3.3; 20]. The number of moduli of those Z obtained by varying X_1 on Y, however, is generally less than the number of moduli of Z itself, hence the remark.

The class of varieties Y to which the hypotheses apply is closed under formation of Cartesian products and includes flag varieties (e.g., projective spaces, quadrics, grassmannians). Weighted projective spaces, however, do not lie in this class, since they are V-manifolds. However, in this case one may use the method of [2] to prove the result directly, along with precise degree conditions (see Appendix). It seems likely that there are analogous theorems for complete intersections in rigid ambient varieties, both in the smooth and in the V-manifold case.

To situate this result we note that there is a dimension estimate for variations of Hodge structure derived from an analysis of the integrability conditions imposed by $\mathcal{G}$ [1, Theorem 1.1; 5, Theorem 1.6]:

$$\dim(\text{integral submanifold}) \leq q(\vec{h}),$$

where q is a quadratic function of the Hodge numbers $\vec{h}$ satisfying

$$q(\vec{h}) \lesssim \tfrac{1}{2} \dim \mathcal{G}.$$

In weight two

$$q(\vec{h}) = h^{2,0}[h^{1,1}/2] + \varepsilon(h^{1,1}),$$

where $[x]$ is the greatest integer in x, $\varepsilon(n)$ is 0 or 1 according to whether n is even or odd, and $h^{2,0}$ is assumed greater than 1. A variation of dimension $q(\vec{h})$ is therefore maximal (see [5, §7] for existence theorems). The converse, however, is false, as the results of [2] as well as those of the present paper show. For a specific example one may take surfaces in $\mathbb{P}^3$ of degree d. The bound $q(\vec{h})$ is $O(d^6)$, while the dimension of the associated integral manifold is given by the number of moduli, which is $O(d^3)$.

In the next section we outline the proof, which relies on the theory of an infinitesimal variation of Hodge structures [4, 3], and, especially, on Donagi's symmetrizer lemma [8]. While the main ideas are as in [2], the more powerful machinery of Koszul cohomology [12, 14] is needed. In particular, we rely heavily on Green's description [11, §1] of the Hodge structure of a hypersurface.

2. Centralizers and symmetrizers

An *infinitesimal variation of Hodge structures* [4,3] is an algebraic object which encodes the data contained in the differential of a period map $\rho\colon U \to \Gamma\backslash D$: consider a polarized Hodge structure H and let $\mathrm{Gr}_F(H)$ be the graded object associated to the Hodge filtration, so that

$$\mathrm{Gr}_F^p \cong H^{p,q},$$

where $p + q = n = \text{weight}(H)$. Let $\text{End}^*(H)$ be the graded ring of endomorphisms of $\text{Gr}_F(H)$ endowed with a Lie algebra structure in a natural way: $[\phi, \phi'] = \phi\phi' - \phi'\phi$. Because of the Riemann bilinear relations, the polarization $\langle\ ,\ \rangle$ descends to $\text{Gr}_F(H)$. Consequently the graded Lie subalgebra of antisymmetric elements is defined:

$$E^*(H) = \{\phi \in \text{End}^*(H) | \langle \phi(x), y\rangle + \langle x, \phi(y)\rangle = 0\}.$$

It contains a nilpotent Lie subalgebra

$$E^-(H) = \bigoplus_{p<0} E^p(H)$$

which can be identified with the holomorphic tangent space to the period domain D at H. The degree -1 component of this algebra, E^{-1}, can then be identified with the space of tangent vectors satisfying Griffiths' infinitesimal period relation.

Now let T be a complex vector space and let $\delta : T \to E^{-1}(H)$ be a linear map with abelian image:

$$[\delta T, \delta T] = 0.$$

Then the object $V(H) = [\delta\colon T \to E^{-1}(H)]$ is an infinitesimal variation of Hodge structure. These arise geometrically as differentials of period mappings and can be calculated from the Kodaira-Spencer class: let T be $H^1(Y, \Theta)$, and let $\delta(\theta)$ be the direct sum of the cup-product maps

$$\delta^p(\theta) : H^q(Y, \Omega^p) \to H^{q-1}(Y, \Omega^{p+1}).$$

The integrability condition $[\delta(\theta), \delta(\theta')] = 0$ is a consequence of standard identities for the cup-product, or alternatively, for the equality of mixed partial derivatives. By definition the dimension of $V(H)$ is the dimension of $\delta(T)$. Sometimes we shall identify T with its image, so that an infinitesimal variation may be viewed as an abelian subalgebra of $E^*(H)$ which is horizontal, i.e., concentrated in degree -1.

Adopting this viewpoint, we say that an infinitesimal variation is *maximal* if T is contained in no larger horizontal subalgebra. Define the *centralizer* of T in E^{-1} to be the subspace

$$\mathscr{Z}(T) = \{\phi \in E^{-1} | [\phi, \alpha] = 0 \text{ for all } \alpha \in T\}.$$

Since $\mathscr{Z}(T)$ contains all infinitesimal variations which contain T, T is maximal if and only if $T = \mathscr{Z}(T)$.

To establish this last equality, i.e., to show that $\dim(T) = \dim \mathscr{Z}(T)$, we use Donagi's *symmetrizer construction* [8], the definitions of which we now recall. Let $B\colon U \times V \to W$ be a bilinear pairing. Construct a new space

$$\sigma(B) = \{\phi \in \text{Hom}(U, V) | B(u, \phi(u')) = B(u', \phi(u)) \text{ for all } u, u' \in U\}.$$

Construct a new pairing $\tau(B)\colon \sigma(B) \times U \to V$ by setting $\tau(B)(\phi, u) = \phi(u)$. Then σ and τ are the *symmetrizer space* and *symmetrizer map* of the pairing B, and the condition $B(u, \phi(u')) = B(u', \phi(u))$ is the *symmetrizer identity.* The symmetrizer space can be defined in purely homological language [9], namely as the left-most space in the exact sequence

$$0 \to \sigma(B) \xrightarrow{d^0} U^* \otimes V \xrightarrow{d^1} \bigwedge^2 U^* \otimes W, \tag{2.1}$$

where d^0 is the natural inclusion and d^1 is the differential given by

$$d^1(\phi) = [u \wedge u' \mapsto B(u, \phi(u')) - B(u', \phi(u))].$$

The plan of the proof can now be stated:

(a) Exhibit a diagram $T \subset \mathscr{Z}(T) \xrightarrow{i} \sigma(T)$, where $\sigma(T)$ is a symmetrizer space associated to the infinitesimal variation with source T.

(b) Show that the map i is injective.

(c) Show that $\dim T = \dim \sigma(T)$.

We begin with a discussion of (a). For a Hodge structure H let

$$m = \max\{p | H^{p,q} \neq 0\}$$

be the maximal Hodge index, and let l be the minimum index, where $n = m+l$ is the weight. For an infinitesimal variation V of H consider the bilinear map

$$B\colon \operatorname{Gr}_F^m(H) \times \operatorname{Gr}_F^{m-1}(H) \to S^k(T^*)$$

defined by

$$B(\omega, \eta)(\phi_1, \ldots, \phi_k) = \langle \omega, \phi_1 \cdots \phi_k \eta \rangle,$$

where S^k denotes the kth symmetric power and where $k = 2m - 1 - n$ is chosen so that the inner product is potentially nonzero.

(2.2) **Lemma.** *For any infinitesimal variation of Hodge structure there is a natural map i of $\mathscr{Z}(T)$ to $\sigma(T)$, where $\sigma(T)$ is the symmetrizer space of*

$$B\colon \operatorname{Gr}_F^m(H) \times \operatorname{Gr}_F^{m-1}(H) \to S^k(T^*).$$

Proof. Since an element of E^{-1} defines a homomorphism of $\operatorname{Gr}^m(H)$ to $\operatorname{Gr}^{m-1}(H)$, it suffices to show that an element ζ of $\mathscr{Z}(T)$ satisfies the symmetrizer identity. Using the skew-symmetry of elements of $E^*(H)$ with respect to $\langle\ ,\ \rangle$, the commutativity properties of elements of T and of $\mathscr{Z}(T)$, the symmetry of $\langle\ ,\ \rangle$, and the fact that $k = 2m - n - 1$, one finds that

$$\begin{aligned}\langle x, \phi_1 \cdots \phi_k \zeta(y) \rangle &= (-1)^{k+1} \langle \zeta \phi_k \cdots \phi_1(x), y \rangle = (-1)^{k+1} \langle \phi_1 \cdots \phi_k \zeta(x), y \rangle \\ &= (-)^{k+1+n} \langle y, \phi_1 \cdots \phi_k \zeta(x) \rangle = \langle y, \phi_1 \cdots \phi_k \zeta(x) \rangle,\end{aligned}$$

which shows $B(x, \zeta(y))(\phi_1 \cdots \phi_k) = B(y, \zeta(x))(\phi_1 \cdots \phi_k)$, as required.

To treat (b) we call an infinitesimal variation *nondegenerate* if it satisfies the following:

(2.3) **Condition.** *The maps* $T \otimes \mathrm{Gr}_F^p \to \mathrm{Gr}_F^{p-1}$ *are surjective for all* p *in the range* $[m, n/2)$, *where* m *is the maximal Hodge index and where* n *is the weight.*

In the presence of this condition the lemma above yields the following result:

(2.4) **Lemma.** *For nondegenerate infinitesimal variations of Hodge structure the natural map* i *of* $\mathscr{Z}(T)$ *to* $\sigma(T)$ *is an injection.*

The lemma implies that $\dim T \le h^{m,n-m} h^{m-1,n-m+1}$ [1, 4.1 and Erratum].

Proof. Let ζ be an element of $\mathscr{Z}(T)$ which annihilates $\mathrm{Gr}_F^m(H)$. We must show that ζ annihilates all of the $\mathrm{Gr}_F^p(H)$. To this end let η be an element of $\mathrm{Gr}_F^p(H)$ with $p \in [m, n/2)$, and let $\omega_i \in \mathrm{Gr}_F^m(H)$, $\Phi_i \in S^k T$ be elements satisfying $\sum_i \Phi_i(\omega_i) = \eta$. This is possible by nondegeneracy. Then $\zeta(\eta) = \sum_i \zeta\Phi_i(\omega) = \Phi_i\zeta(\omega_i) = 0$.

To treat (c) let $H^1(Z, \Theta)_0$ be the subspace of $H^1(Z, \Theta)_0$ generated by infinitesimal variations coming from $|L|_0$. If the differential of the period mapping is injective then T can be identified with $H^1(Z, \Theta)_0$. This is the *local Torelli assumption*, in the presence of which the condition reduces to $\dim H^1(Z, \Theta)_0 = \dim \sigma(T)$. Consequently one has the following maximality criterion:

(2.5) **Proposition.** *A family of varieties containing* Z *defines a maximal variation of Hodge structures if the following hold for the infinitesimal variation of* $H^m(Z)_0$:

(i) *nondegeneracy,*
(ii) *local Torelli,*
(iii) $\dim H^1(Z, \Theta)_0 = \dim \sigma(T)$.

As we shall see in §8, sufficiently ample subvarieties define nondegenerate variations, provided that the ambient variety has first Betti number zero. Moreover, as Green showed using Koszul methods [11, Theorem 0.1], sufficiently ample hypersurfaces in an ambient variety of dimension three or more satisfy local Torelli. In the next section we formulate (iii), which seems the most refractory component of the hypothesis, in terms of Koszul cohomology. The proof of Theorem (1.1) is thereby reduced to proving that certain Koszul groups vanish.

3. Koszul cohomology

We begin by recalling the Lieberman-Peters treatment of Koszul cohomology methods [14]. Consider a line bundle W, a nonzero subspace $V \subset H^0(Z, W)$, and observe that the vector bundle $E = W^* \otimes_{\mathbb{C}} V$ admits a map $D: E \to \mathscr{O}_Z$ defined by $\phi \otimes \sigma \mapsto \phi(\sigma)$. Extend D as a derivation on $\bigwedge^i E$, one to obtain the Koszul complex

$$\bigwedge^* E = [\bigwedge^q E \to \cdots \to \bigwedge^2 E \to E \to \mathscr{O}].$$

Given a vector bundle F, set

$$K_{WV}(F) = \mathrm{Hom}_{\mathcal{O}}(\bigwedge^* E, F), \tag{3.1}$$

or, in more detail,

$$K_{WV}(F) = \left[F \to V^* \otimes F(W) \to \bigwedge^2 V^* \otimes F(2W) \to \cdots\right].$$

When the context makes the meaning clear we omit one or more of the subscripts. Now let

$$K(H^q(Z, F)) \underset{\text{def}}{=} H^q(Z, K(F))$$

be the complex of vector spaces obtained by applying the functor H^q to $K(F)$. Define, as in [14, 2.3], the Koszul cohomology by

$$H^p_{\mathrm{Koz}}(H^q(Z, F)) = H^p(K(H^q(Z, F))).$$

When the context is ambiguous, we may augment the subscript, e.g., $H^p_{\mathrm{Koz}\,WV}$.

Consider now the case in which $W = \Omega^m$, $V = H^0(\Omega^m)$, and $F = \Theta$, where $m = \dim Z$. Then $K_{\Omega^m}(H^1(\Theta))$ is

$$H^1(\Theta) \xrightarrow{d^0} H^0(\Omega^m)^* \otimes H^1(\Theta \otimes \Omega^m) \xrightarrow{d^1} \bigwedge^2 H^0(\Omega^m)^* \otimes H^1(\Theta \otimes \Omega^{m\otimes 2}) \to \cdots \tag{3.2}$$

which can be rewritten as

$$H^1(\Theta) \xrightarrow{d^0} H^0(\Omega^m)^* \otimes H^1(\Omega^{m-1}) \xrightarrow{d^1} \bigwedge^2 H^0(\Omega^m)^* \otimes H^1(\Omega^m \otimes \Omega^{m-1}) \to \cdots.$$

Apply Serre duality to the last factor of the last term to obtain

$$H^1(\Theta) \xrightarrow{d^0} H^0(\Omega^m)^* \otimes H^1(\Omega^{m-1}) \xrightarrow{d^1} \bigwedge^2 H^0(\Omega^m)^* \otimes H^{m-1}(\bigwedge^{m-1}\Theta)^* \to \cdots. \tag{3.3}$$

The map d^0 is the differential of the period map, so that, as was shown in [14, p. 41], local Torelli is equivalent to the vanishing of the group $H^0_{\mathrm{Koz}\,\Omega^m}(H^1(\Theta))$. Now consider the bilinear pairing of Lemma (2.2):

$$B\colon H^0(\Omega^m) \times H^0(\Omega^{m-1}) \to S^{m-1}H^1(\Theta)^*.$$

Because the expression

$$B(\omega, \nu)(\phi_1, \ldots, \phi_{m-1}) = \langle \omega, \phi_1 \cdots \phi_{m-1}\nu\rangle$$

depends only on the value of $\phi_1 \cdots \phi_{m-1}$ in $H^{m-1}(\bigwedge^{m-1}\Theta)$, the symmetrizer space of B is equal to the symmetrizer space of the pairing

$$B'\colon H^0(\Omega^m) \times H^0(\Omega^{m-1}) \to H^{m-1}(\bigwedge^{m-1}\Theta)^*,$$

i.e., is the kernel of d^1. Thus, if the zeroth Koszul group in (3.2) vanishes, then condition (iii) in criterion (2.5) is equivalent to vanishing of the first Koszul group. We therefore have a cohomological criterion for maximality of a geometric variation:

(3.4) **Theorem.** *Let* Z *be a smooth projective variety of dimension* m *for which the following hold*:

(i) *The map* $H^1(\Theta)\otimes H^q(\Omega^p)\to H^{q+1}(\Omega^{p-1})$ *is surjective.*

(ii) *The Koszul groups* $H^0_{\mathrm{Koz}\,\Omega^m}(H^1(\Theta))$ *and* $H^1_{\mathrm{Koz}\,\Omega^m}(H^1(\Theta))$ *vanish.*

Then the versal family in which Z *lies defines, by* $Z\mapsto H^m(Z)$, *a maximal variation of Hodge structure.*

Verification of the hypotheses of the preceding theorem seems difficult in general. However, under the hypotheses of Theorem (1.1) this verification can be carried out. To do so it is necessary to first replace the sequence (3.2) by one for the primitive cohomology of Z, defined here as the kernel of the Gysin map, and then to transform it to an equivalent sequence in which all terms are calculated in terms of zeroth rather than higher cohomology. We sketch here how this is done, with the details to be given in the body of the paper. To begin, recall [3, XX] that the *Jacobian system* of a coherent sheaf $\mathscr{F}$ on Y is defined by

$$JH^k(\mathscr{F}) \underset{\text{def}}{=} \operatorname{image}\{H^k(\mathscr{F}\otimes\Theta_Y(-Z)\otimes\mathscr{O}_Z)\to H^k(\mathscr{F}\otimes\mathscr{O}_Z)\},$$

with *Jacobian quotient*

$$H^k(\mathscr{F})_J \underset{\text{def}}{=} \operatorname{cokernel}\{H^k(\mathscr{F}\otimes\Theta_Y(-Z)\otimes\mathscr{O}_Z)\to H^k(\mathscr{F}\otimes\mathscr{O}_Z)\}.$$

Denote the primitive cohomology by $H^*(Z)_0$, and let $H^1(\Theta_Z)_0$ denote the kernel of $H^1(\Theta_Z)\to H^1(\Theta_Y)$. From the normal bundle sequence

$$0\to\Theta_Z\to\Theta_Y\otimes\mathscr{O}_Z\to N_Z\to 0 \tag{3.5}$$

one obtains the usual identification

$$H^1(\Theta_Z)_0\cong H^0(N_Z)_J. \tag{3.6}$$

From a Koszul complex constructed from the normal bundle sequence one obtains the identification.

$$H^p(\Omega^q_Z)_0\cong H^0(\Omega^m_Z\otimes N^q_Z)_J, \tag{3.7}$$

provided that $Z\gg 0$ and that $b_1(Y)=0$. In addition, we find that

$$H^m(\Omega^m_Z)\cong H^0(\Omega^{m\otimes 2}_Z\otimes N^m_Z)_J, \tag{3.8}$$

so that the cup-product on primitive cohomology can be identified with the pairing

$$H^0(\Omega^m_Z\otimes N^q_Z)_J\times H^0(\Omega^m_Z\otimes N^p_Z)_J\to H^0(\Omega^{m\otimes 2}_Z\otimes N^m_Z)_J, \tag{3.9}$$

where $p+q=m$. One has also the pairings

$$H^0(N^q_Z)_J\times H^0(\Omega^{m\otimes 2}_Z\otimes N^p_Z)_J\to H^0(\Omega^{m\otimes 2}_Z\otimes N^m_Z)_J. \tag{3.10}$$

We cannot, however, show that these latter are perfect except in special cases, e.g., $q=1$ with rigid ambient space, i.e., $H^1(\Theta_Y)=0$. By the remark following Theorem (1.1), this lack of perfection is real.

Given the preceding apparatus, and assuming (a) $Z \gg 0$ and (b) $b_1(Y) = 0$, we may write the analogue of the sequence (3.3) as

(3.11) $H^0(N_Z)_J \to H^0(\Omega_Z^m)_0^* \otimes H^0(\Omega_Z^m \otimes N_Z)_J \to \bigwedge^2 H^0(\Omega_Z^m)_0^* \otimes H^0(N_Z^{m-1})_J^*$.

Under the additional assumption (c) that Y is rigid, the dual of the preceding complex can be written as

$$(3.12)\qquad H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1})_J \leftarrow H^0(\Omega_Z^m)_0 \otimes H^0(\Omega_Z^m \otimes N_Z^{m-1})_J \leftarrow \bigwedge^2 H^0(\Omega_Z^m)_0 \otimes H^0(N_Z^{m-1})_J.$$

This is a quotient of the Koszul complex $K_{WV}(F)$, where $W = \Omega_Z^m$, $V = H^0(\Omega_Z^m)_0$, and $F = N$. The analogue of Theorem (3.4) is therefore

(3.13) **Proposition.** *A hypersurface variation defines a maximal variation on the primitive cohomology if*

(i) *it is nondegenerate, and*

(ii) $H^i(K_{\Omega_Z^m|\Omega_Z^m|_0}(H^0(N_Z))_J) = 0$ *for* $i = 0, 1$.

4. Hodge theory for hypersurfaces

The purpose of this section is to establish the identification

$$(4.1)\qquad H^q(\Omega_Z^p) \cong \mathbb{H}^0(\Omega_Z^m \otimes K_q(\pi)),$$

where $Z \subset Y$ is a submanifold of arbitrary codimension and $K_q(\pi)$ is a Koszul complex derived from the normal bundle sequence. This is a restatement of [9, 1.5], whose proof in fact gives something stronger, namely a quasi-isomorphism

$$(4.2)\qquad \Omega_Z^p[q] \cong K_q(\pi),$$

where $K[q]$, as usual, denotes the complex K shifted q positions to the left. For sufficiently ample hypersurfaces Z the right-hand group of (4.1) will reduce, as we see in §7, to $H^0(\Omega_Z^m \otimes N_Z^q)_J$.

We begin with some purely homological preliminaries. Write a complex $K = [A \to B \to \cdots \to R \to S]$ as $[p; A \to B \to \cdots \to R \to S]$ or $[A \to B \to \cdots \to R \to S; q]$ to make the absolute degrees of the terms explicit, where p and q are the degrees of the left- and right-most terms, respectively. An exact sequence $0 \to A \to F_0 \to F_1 \to \cdots \to F_n$ defines a morphism of complexes, $A \cong [0; F_0 \to F_1 \to \cdots \to F_n]$, which induces an isomorphism on hypercohomology, i.e., is an isomorphism in the derived category. Similar considerations apply to $F_0 \to F_1 \to \cdots \to F_n \to B \to 0$, which yields the quasi-isomorphism $[F_0 \to F_1 \to \cdots \to F_n; 0] \cong B$. Consider next a short complex

$$K = [A \xrightarrow{\phi} B; 0]$$

and form its *qth Koszul complex,*

$$K_q(\phi) = [\bigwedge^q A \to \cdots \to \bigwedge^{q-i} A \otimes S^i B \to \cdots \to S^q B; 0],$$

where S^i denotes the ith symmetric power.

(4.3) **Lemma.** *In the category of vector bundles over a fixed manifold,* $K_q(\phi)$ *is exact if* ϕ *is either injective or surjective.*

Proof. It suffices to verify the lemma for the case in which the base manifold is a point, i.e., to verify it in the category of vector spaces. To begin, note that the differential in the Koszul complex is given by

$$\delta(e_{i_1}\cdots e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_q})=\sum e_{i_1}\cdots\tilde{e}_{i_a}\cdots\tilde{e}_{i_p}\otimes\phi(e_{i_a})\wedge f_{j_1}\wedge\cdots\wedge f_{j_q}.$$

One may choose bases $\{e_1, \dots, e_m\}$ for A and $\{f_1, \dots, f_n\}$ for B so that $\phi(e_i) = f_i$ for $i \le r$ and $\phi(e_i) = 0$ for $i > r$, where r is the rank. Define a function on indices by $\varepsilon(i) = 1$ if $i \le r$, $\varepsilon(i) = 0$ if $i > r$. Then

$$\delta(e_{i_1}\cdots e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_q})=\sum\varepsilon(i_a)e_{i_1}\cdots\tilde{e}_{i_a}\cdots e_{i_p}\otimes e_{i_a}\wedge f_{j_1}\wedge\cdots\wedge f_{j_q}.$$

Define a pseudo-homotopy operator by

$$\begin{aligned}&H(e_{i_1}\cdots e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_q})\\&\quad=\sum\varepsilon(j_b)(-1)^{j_b}e_{i_1}\cdots e_{i_p}e_{j_b}\otimes f_{j_1}\wedge\cdots\wedge\tilde{f}_{j_b}\wedge\cdots\wedge f_{j_q}.\end{aligned}$$

For a multi-index I, let $e(I)$ denote the number of indices less than or equal to r and let $e(I, J) = e(I) + e(J)$. One verifies that

$$(dH+Hd)(e_{i_1}\cdots e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_q})=\varepsilon(I, J)(e_{i_1}\cdots e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_q}).$$

If ϕ is either injective or surjective, then $\varepsilon(I, J)$ is positive for all pairs (I, J), so that $dH+Hd$ is a nonsingular operator K. But then the complex in question is exact, for if $dx = 0$, then $dHx = Kx$, and so $x = dHK^{-1}x$.

The preceding lemma allows one to "unroll" the kernel or cokernel of a morphism of maximal rank, i.e., to replace it by a quasi-isomorphic cochain complex:

(4.4) **Corollary.** *Consider a short exact sequence of vector bundles*

$$0\to A\xrightarrow{\phi}B\xrightarrow{\psi}C\to 0.$$

Then there are quasi-isomorphisms $\bigwedge^q A \cong K_q(\psi)[-q]$ *and* $S^q B \cong K_q(\phi)$.

(4.5) *Remark.* Truncation defines a natural increasing filtration of $F \otimes K_q(\phi)$ by subcomplexes

$$\beta_r(F\otimes K_q(\phi))=[\textstyle\bigwedge^r A\otimes S^{q-r}B\to\cdots\to S^q B;0].$$

We write $\beta^r = \beta_{-r}$ for the corresponding decreasing filtration (the filtration bête).

The quasi-isomorphism (4.2) follows from the construction of (4.4) applied to the normal bundle sequence of Z in Y,

$$0\to\Theta_Z\to\Theta_Y\otimes\mathscr{O}_Z\xrightarrow{\pi}N_Z\to 0.$$

Indeed, we have

$$\Theta_Z[1] \cong [\Theta_Y \otimes \mathscr{O}_Z \xrightarrow{\pi} N_Z\,;\,0] \underset{\text{def}}{=} K_1(\pi)\,, \tag{4.6}$$

so that

$$\textstyle\bigwedge^q \Theta_Z[q] \cong K_q(\pi)\,, \tag{4.7}$$

where

$$K_q(\pi) = \left[\textstyle\bigwedge^q \Theta_Y \otimes \mathscr{O}_Z \to \cdots \to \bigwedge^{q-i}\Theta_Y \otimes N_Z^i \to \cdots \to N_Z^q\,;\,0\right]. \tag{4.8}$$

Tensor this isomorphism with the canonical bundle of Z to obtain

$$\Omega_Z^p[q] \cong \Omega_Z^m \otimes \textstyle\bigwedge^q \Theta_Z[1] \cong \Omega_Z^m \otimes K_q(\pi)\,,$$

hence the required identification. The remaining apparatus of the infinitesimal variation of Hodge structure for Z can now be defined via natural pairings on the level of complexes of sheaves. For the differential of the period mapping δ one uses

$$\Theta_Z[1] \times \Omega_Z^p[q] \to \Omega_Z^{p-1}[q+1] \tag{4.9}$$

which becomes

$$K_1(\pi) \otimes (\Omega_Z^m \otimes K_q(\pi)) \to \Omega_Z^m \otimes K_{q+1}(\pi). \tag{4.10}$$

For the cup-product we note that

$$\Omega_Z^m[m] \cong \textstyle\bigwedge^m \Theta_Z[m] \otimes (\Omega_Z^m)^{\otimes 2} \cong K_m(\pi) \otimes (\Omega_Z^m)^{\otimes 2}\,, \tag{4.11}$$

so that the pairing

$$\Omega_Z^p[q] \times \Omega_Z^q[p] \to \Omega_Z^m[m] \tag{4.12}$$

becomes

$$(\Omega_Z^m \otimes K_q(\pi)) \times (\Omega_Z^m \otimes K_p(\pi)) \to \Omega_Z^{m\otimes 2} \otimes K_m(\pi). \tag{4.13}$$

5. Gysin maps

As remarked above, the presentation $\Omega_Z^p[q] \cong \Omega_Z^m \otimes K_q(\pi)$ defines a "Koszul" filtration $\beta^0 \subset \beta^{-1} \subset \cdots \subset \beta^{-q}$ of $H^*(Z\,,\Omega^p)$. For the case of hypersurfaces we relate this filtration to the Gysin map:

(5.1) **Proposition.** *The sequence below is exact*:

$$0 \to \beta^{1-q} H^1(\Omega_Z^p) \to H^i(\Omega_Z^p) \overset{Gysin}{\to} H^{i+1}(\Omega_Y^{p+1}).$$

With the obvious changes of the index shift, the same result holds for arbitrary codimension.

Proof. We use the derived category to

(a) represent the Gysin homomorphism by a map

$$\Omega_Z^p[q] \xrightarrow{\gamma_1} \Omega_Y^{p+1}[q+1],$$

(b) construct a map

$$\Omega_Z^m \otimes K_q(\pi) \xrightarrow{\gamma_2} \Omega_Y^{p+1}[q+1]$$

whose kernel is β^{1-q},

(c) prove that the natural diagram relating α and β is commutative,

$$\begin{array}{ccc} \Omega_Z^p[q] & \to & \Omega_Y^{p+1}[q+1] \\ \downarrow & \nearrow & \\ \Omega_Z^m \otimes K_q(\pi) & & \end{array}$$

where the vertical arrow is an isomorphism.

For part (a) note that the Gysin map is given by the coboundary in the Poincaré residue sequence,

$$0 \to \Omega_Y^* \to \Omega_Y^*(\log Z) \to \Omega_Z^* \to 0,$$

which can be rewritten as

$$(*) \qquad \Omega_Z^*[-1] \cong [\Omega_Y^* \to \Omega_Y^*(\log Z)\,;\, 0] \to \Omega_Y^*[1],$$

or more simply as

$$\Omega_Z^* \to \Omega_Y^*[2].$$

The basic construction $(*)$, formation of the mapping cone of the inclusion of the de Rham in the log complex, is compatible with the Hodge filtration, here given by the "filtration bête", so that on the graded quotients we find

$$\gamma_1 : \Omega_Z^p \cong [\Omega_Y^{p+1} \to \Omega_Y^{p+1}(\log Z)\,;\, 0] \to \Omega_Y^{p+1}[1].$$

A second possible incarnation of the Gysin maps comes from the presentation of Ω_Z^p as $\Omega_Z^m \otimes K_q(\pi)[-q]$, namely, from projection to the quotient complex:

$$\begin{aligned} \gamma_2 : \Omega_Z^m \otimes K_q(\pi)[-q] &\to \Omega_Z^m \otimes \textstyle\bigwedge^q \Theta_Y \\ &\cong [\Omega_Y^n \otimes \textstyle\bigwedge^q \Theta_Y \to \Omega_Y^n(Z) \otimes \textstyle\bigwedge^q \Theta_Y\,;\, 0] \to \Omega_Y^n \otimes \textstyle\bigwedge^q \Theta_Y[1] \cong \Omega_Y^{p+1}[1]. \end{aligned}$$

To prove that γ_1 and γ_2 are related as asserted (part (c)), it suffices to show that the diagram below is commutative:

$$\begin{array}{ccccc} \Omega_Z^p & \overset{\cong}{\leftarrow} & [\Omega_Y^{p+1} \to \Omega_Y^{p+1}(\log Z)\,;\, 0] & \to & \Omega_Y^{p+1}[1] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \Omega_Z^m \otimes K_q(\pi)[-q] & \leftarrow & [\Omega_Y^n \otimes \bigwedge^q \Theta_Y \to \Omega_Y^n(\log Z) \otimes \bigwedge^q \Theta_Y\,;\, 0] & \to & \Omega_Y^n \otimes \bigwedge^q \Theta_Y[1] \end{array}$$

To this end consider an element

$$\eta = A\,dz_I[1] + B\,dz_J \wedge \frac{dz_n}{z_n} + C\,dz_K$$

of the upper middle term, where $\alpha[k]$ denotes an element (germ, local section, etc.) shifted k units to the left, i.e., taken in degree k. Following η to the left and then down, we find

$$\eta \mapsto B dz_J \mapsto B\omega \otimes \left(\frac{\partial}{\partial z}\right)_{J^\circ},$$

where $\omega = dz_1 \wedge \cdots \wedge dz_m$ and J° is the index "opposite" to J:

$$J \cup J^\circ = \{1, \ldots, m\},$$

with J° ordered in the natural way. Following η down and to the left, we find

$$\begin{aligned}\eta &\mapsto A\omega \wedge dz_n \otimes \left(\frac{\partial}{\partial z}\right)_{I^\circ}[1] + B\omega \wedge \frac{dz_n}{z_n} \otimes \left(\frac{\partial}{\partial z}\right)_{J^\circ} \\ &+ C\omega \wedge dz_n \otimes \left(\frac{\partial}{\partial z}\right)_{K^\circ} \\ &\mapsto B\omega \otimes \left(\frac{\partial}{\partial z}\right)_{J^\circ},\end{aligned}$$

where $I \cup I^\circ = \{1, \ldots, n\}$. This establishes commutativity of one panel of the diagram. The argument for the other panel is similar.

6. The very ample case

In the very ample case the Koszul filtration β has a simple description:

(6.1) **Proposition.** *Let Z be a sufficiently ample hypersurface. Then $H^q(Z, \Omega^p)$ is concentrated in levels 0 and $-q$ of the Koszul filtration, with*

$$\beta^0 H^q(\Omega^p) = \text{kernel}\{\mathit{Gysin}\colon H^q(Z, \Omega^p) \to H^{q+1}(Y, \Omega^{p+1})\} \underset{\text{def}}{=} H^q(Z, \Omega^p)_0.$$

Proof. Since $\Omega^p_Z[q] \cong \Omega^m_Z \otimes K_q(\pi)$,

$$H^q(\Omega^p_Z) = \mathbb{H}^0(\Omega^p_Z[q]) = \mathbb{H}^0(\Omega^m_Z \otimes K_q(\pi)).$$

The proof results from an analysis of the spectral sequence of hypercohomology associated to the Koszul filtration. The E_1 term is

$$E_1^{-r,s} = H^s(\Omega^m_Z \otimes \textstyle\bigwedge^r \Theta_Y \otimes N_Z^{q-r}),$$

which we can analyze using the Poincaré residue sequence:

$$H^s(\Omega^n_Y((q-r+1)Z) \otimes \textstyle\bigwedge^r \Theta_Y) \to E_1^{-r,s} \to H^{s+1}(\Omega^n_Y((q-r)Z) \otimes \textstyle\bigwedge^r \Theta_Y).$$

By the Kodaira vanishing theorem, the middle term vanishes unless $s = 0$ or $r = q$, so that the only contributions to the term of total degree 0 are from $E_1^{0,0}$ and $E_1^{-q,q}$. Moreover, since all differentials emanating from $E_r^{-q,q}$ are zero,

$$\mathrm{Gr}_\beta^{-q} H^q(\Omega^p_Z) = E_\infty^{-q,q} = E_1^{-q,q} = H^q(\Omega^m_Z \otimes \textstyle\bigwedge^q \Theta_Y \otimes \mathcal{O}_Z).$$

To identify the right-hand side, apply the Poincaré residue sequence once again to get

$$H^q(\Omega^n_Y(Z) \otimes \textstyle\bigwedge^q \Theta_Y) \to H^q(\Omega^m_Z \otimes \textstyle\bigwedge^q \Theta_Y \otimes \mathscr{O}_Z)$$
$$\to H^{q+1}(\Omega^n_Y \otimes \textstyle\bigwedge^q \Theta_Y) \to H^{q+1}(\Omega^m_Y(Z) \otimes \textstyle\bigwedge^q \Theta_Y).$$

The end terms vanish and the penultimate term is isomorphic to $H^{q+1}(\Omega^{p+1}_Y)$. Therefore the quotient of filtration level $-q$ is identified with the image of $H^q(Z, \Omega^p)$ under the Gysin map, and this concludes the proof.

Further study of the level zero term yields the formula (3.8):

(6.2) **Proposition.** *There is a canonical identification*

$$H^q(\Omega^p_Z)_0 \cong H^0(\Omega^m_Z \otimes N^q_Z)_J.$$

Proof. From the definitions we find

$$E^{0,0}_2 = \text{cokernel}\{H^0(\Omega^m_Z \otimes \Theta_Y \otimes N^{q-1}_Z) \to H^0(\Omega^m_Z \otimes N^q_Z)\} = H^0(\Omega^m_Z \otimes N^q_Z)_J.$$

Therefore we must show that $E^{0,0}_2 = E^{0,0}_\infty$. The only differential with target in an iterated subquotient of $E^{0,0}_2$ is $d_q \colon E^{-q,q-1}_q \to E^{0,0}_q$. Now, on the one hand, $E^{-q,q-1}_q$ has an iterated subquotient which occurs as part of the β-graded object associated to $\mathbb{H}^{-1}(\Omega^p_Z[q]) \cong H^{p,q-1}(Z)$. On the other hand, $E^{-q,q-1}_q$ can be identified with $H^{p+1,q}(Y)$:

$$E^{-q,q-1}_q = E^{-q,q-1}_1 = H^{q-1}(\Omega^m_Z \otimes \textstyle\bigwedge^q \Theta_Y) \cong H^q(\Omega^n_Y \otimes \textstyle\bigwedge^q \Theta_Y) \cong H^{p+1,q}(Y),$$

where the penultimate isomorphism comes from the Poincaré residue sequence. Because Z is very ample the Gysin map gives an isomorphism of $H^{p,q-1}(Z)$ with $H^{p+1,q}(Y)$. Therefore $E^{-q,q-1}_q = E^{-q,q-1}_\infty$, and so d_q must vanish. Consequently $E^{0,0}_2 = E^{0,0}_\infty$, and so the proof is complete.

(6.3) *Remark.* Write $H^1(\Theta_Z)_0$ for the β^0 subspace of $H^1(\Theta_Z)_0$. Then

$$H^1(\Theta_Z)_0 = \beta^0 H^1(\Theta_Z) \cong H^0(N_Z)_J.$$

Note that $H^0(N_Z)_J$ is the space of infinitesimal deformations arising from $|L|_0$.

7. Duality

The aim of this section is to establish the following duality theorem.

(7.1) **Theorem.** *Let Z be a sufficiently ample smooth hypersurface of a variety Y, where $H_1(Y, \mathbb{Q}) = 0$. Let $m = \dim Z$. Then the vector space $H^0(\Omega^{m\otimes 2}_Z \otimes N^m_Z)_J$ is one dimensional and the pairings*

$$H^0(\Omega^m_Z \otimes N^i_Z)_J \times H^0(\Omega^m_Z \otimes N^{m-i}_Z)_J \to H^0(\Omega^{m\otimes 2}_Z \otimes N^m_Z)_J$$

are perfect for $i = 0, \ldots, m$. *If in addition* $H^1(\Theta_Y) = 0$, *then the pairing*

$$H^0(N_Z^i)_J \times H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-i})_J \to H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^m)_J$$

is perfect for $i = 1$.

A. *Proof that* $H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^m)_J$ *is one dimensional.* We begin with the observation that

$$\begin{aligned}\Omega_Z^m[m] &\cong \Omega_Z^{m\otimes 2} \otimes K_m(\pi) \\ &= \Omega_Z^{m\otimes 2} \otimes \left[\bigwedge^m \Theta_Y \otimes \mathcal{O}_Y \to \bigwedge^{m-1}\Theta_Y \otimes N_Z \to \cdots \to N_Z^m ; 0\right].\end{aligned}$$

Thus,

$$\mathbb{C} \cong H^m(\Omega_Z^m) = \mathbb{H}^0(\Omega_Z^m[m]) = \mathbb{H}^0(\Omega_Z^{m\otimes 2} \otimes K_m(\pi)).$$

The spectral sequence for hypercohomology gives

$$\begin{aligned} E_1^{0,0} &= H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^m), \\ E_1^{-1,0} &= H^0(\Omega_Z^{m\otimes 2} \otimes \Theta_Y \otimes N_Z^{m-1}),\end{aligned}$$

so that

$$E^{0,0} = H^0(N_Z^m)_J.$$

It therefore remains to show that $E_2^{0,0} = E_\infty^{0,0}$, and that $E_\infty^0 = E_\infty^{0,0}$. To this end, note that the differentials with target $E_r^{0,0}$ emanate from $E_r^{-r,r-1}$, which is an iterated subquotient of

$$E_1^{-r,r-1} = H^{r-1}\left(\Omega_Z^{m\otimes 2} \otimes \bigwedge^r \Theta_Y \otimes N_Z^{m-r}\right).$$

Apply the adjunction formula to obtain $\Omega_Z^{m\otimes 2} \cong \Omega_Y^{n\otimes 2}(2) \otimes \mathcal{O}_Z$ and then the Poincaré residue sequence to get

$$0 \to \Omega_Y^{n\otimes 2}(1) \to \Omega_Y^{n\otimes 2}(2) \to \Omega_Z^{m\otimes 2} \to 0,$$

where $\mathcal{O}_Y(1) =_{\text{def}} \mathcal{O}_Y(Z)$. Apply the Kodaira vanishing theorem to the associated long exact cohomology sequence to conclude that $E_1^{-r,r-1} = 0$ for $r > 1$ and hence that $E_2^{0,0} = E_\infty^{0,0}$. Arguing in the same way, one finds that $E_1^{-r,r} = 0$ for $r > 0$, so that $E_\infty^0 = E_\infty^{0,0}$.

B. *Proof that the first pairing is perfect.* This is immediate from the fact that (a) the pairing is the topological cup-product on primitive cohomology and (b) the primitive cohomology is a sub-Hodge structure.

C. *Proof that the second pairing is perfect.* Tensor the mth Koszul complex of $\pi : \Theta_Y \otimes \mathcal{O}_Z \to N_Z$ with $\Omega_Z^{m\otimes 2} \otimes N_Z^{-1}$ to obtain an exact sequence

$$0 \to \Omega_Z^m \otimes N_Z^{-1} \to \Omega_Z^{m\otimes 2} \otimes \bigwedge \Theta_Y \otimes N_Z^{-1} \to \cdots \to \Omega_Z^{m\otimes 2} \otimes N_Z^{m-1} \to 0.$$

The spectral sequence of the resulting complex has among its terms

$$E_1^{0,0} = H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1}),$$
$$E_1^{-m-1,m} = H^m(\Omega_Z^{m\otimes 2} \otimes N_Z^{-1}) \cong H^0(N_Z)^*.$$

Since the abutment is zero, the differential on the corresponding terms at level $-m-1$,

$$d_{m+1}\colon E_{m+1}^{-m-1,m} \to E_{m+1}^{0,0},$$

must be an isomorphism. But the terms at level 2 are easily computed to be

$$E_2^{0,0} = H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1})_J \quad \text{and} \quad E_2^{-m-1,m} \cong H^0(N_Z)_J^*,$$

so the required duality will follow if these coincide with the corresponding E_{m+1} terms. Now the potential targets of differentials with source in $E_r^{-m-1,m}$ for $1 < r < m+1$ are subquotients of

$$E_1^{r-m-1,m-r+1} = H^{m-r+1}\left(\Omega_Z^{m\otimes 2} \otimes \bigwedge^{m+1-r}\Theta_Y \otimes N_Z^{r-2}\right).$$

Applying the adjunction formula, the Poincaré residue sequence, and the Kodaira theorem as above, one concludes that these groups are zero if $1 < r < m+1$. Therefore $E_2^{-m-1,m} = E_{m+1}^{-m-1,m}$. Similarly, the potential sources of differentials with targets in $E_r^{0,0}$ for $1 < r < m$ are subquotients of

$$E_1^{-r-1,r} = H^r\left(\Omega_Z^{m\otimes 2} \otimes \bigwedge^{r+1}\Theta_Y \otimes N_Z^{m-r-2}\right).$$

Applying the standard argument, one finds this group to be zero for $1 < r < m-1$. For $r = m-1$, however, one finds that

$$E_1^{m,m-1} \cong H^1(\Theta_Y)^*.$$

If this group is zero, then the required conditions hold, and so the duality is established.

8. Nondegeneracy

We can now show that sufficiently ample hypersurface variations are nondegenerate.

(8.1) **Proposition.** *Let L be a positive line bundle on a smooth projective variety Y with vanishing first Betti number. Then there is a k_0 such that for all $k \geq k_0$ and all $Z \in |L^k|_0$, the infinitesimal variation defined by $Z \mapsto H^m(Z)_0$ is nondegenerate.*

Proof. The infinitesimal variation of Hodge structure in question is given by the pairings

$$H^1(\Theta_Z)_0 \times H^q(\Omega_Z^p)_0 \to H^{q+1}(\Omega^{p-1})_0,$$

which, by the discussion of the preceding sections, can be identified with

$$H^0(N_Z)_J \times H^0(\Omega_Z^m \otimes N_Z^q)_J \to H^0(\Omega_Z^m \otimes N_Z^{q+1})_J.$$

Thus, it suffices to show that the map

$$H^0(N_Z) \otimes H^0(\Omega_Z^m \otimes N_Z^q) \to H^0(\Omega_Z^m \otimes N_Z^{q+1})$$

is surjective. From the sequence $0 \to \mathcal{O}_Y \to L^k \to N_Z \to 0$ we obtain

$$H^0(L^k) \to H^0(N_Z) \to H^1(\mathcal{O}_Y),$$

so that by the topological hypothesis, $H^0(L^k) \to H^0(N_Z)$ is surjective. From the Poincaré residue sequence,

$$0 \to \Omega_Y^n \otimes L^{qk} \to \Omega_Y^n \otimes L^{(q+1)k} \to \Omega_Z^m \otimes N_Z^q \to 0,$$

and the Kodaira vanishing theorem we conclude that

$$H^0(\Omega_Y^n \otimes L^{(q+1)k}) \to H^0(\Omega_Z^m \otimes N_Z^q)$$

is surjective for $k \gg 0$. Thus, it suffices to show that the map below is surjective:

$$H^0(L^k) \otimes H^0(\Omega_Y^n \otimes L^{(q+1)k}) \to H^0(\Omega_Y^n \otimes L^{(q+2)k}).$$

But this follows from the result below [11, Lemma 1.28].

(8.2) **Lemma.** *Let Y be a smooth projective variety of dimension n, and let E_1, E_2 be holomorphic vector bundles thereon. For a sufficiently ample line bundle L on Y the multiplication map*

$$H^0(E_1 \otimes L^a) \otimes H^0(E_2 \otimes L^b) \to H^0(E_1 \otimes E_2 \otimes L^{a+b})$$

is surjective for $a \geq 1$ and $b \geq 1$.

One can also give a proof along the lines of [15, 2.6]: Denote $E_i \otimes L^j$ by $E_i(j)$, and let $f: Y \to \mathbb{P}^N$ be the map defined by the linear system $|L|$. Take L ample enough so that $\dim f(Y) = \dim Y$, and let $E_i' = f_* E_i$. Then

$$(*) \qquad H^0(Y, E_i(j)) = H^0(\mathbb{P}^n, E_i'(j)).$$

Since E_i' is coherent, it is, by the results of [17], the quotient of a sheaf E_i'' of the form $\sum_k \mathcal{O}(r_k)$. But the map

$$H^0(\mathbb{P}^n, E_1''(j)) \otimes H^0(\mathbb{P}^n, E_2''(k)) \to H^0(\mathbb{P}^n, E_1'' \otimes E_2''(j+k))$$

is given by multiplication of homogeneous polynomials and so is surjective. This, however, gives surjectivity with E' in place of E'', which, in view of the isomorphism $(*)$, gives the result in question.

9. Completion of the proof

We shall now complete the proof of the main theorem using Theorem (3.4). Part (i), nondegeneracy, was established in the preceding section. For parts (ii) and (iii) we observe that the duality Theorem (7.1) applies so that it remains only to show that the Koszul cohomology groups of (3.12) in degrees 0 and 1 vanish. The vanishing in degree one is Green's result [11, Theorem 0.1].

To begin the computation, we note that $Z \gg 0$ implies that $H^0(\Omega_Z^m)_0$ is basepoint free, so that the underlying complex of sheaves

$$(9.1)\quad \Omega_Z^{m\otimes 2} \otimes N_Z^{m-1} \leftarrow H^0(\Omega_Z^m)_0 \otimes \Omega_Z^m \otimes N_Z^{m-1} \leftarrow \bigwedge^2 H^0(\Omega_Z^m)_0 \otimes N_Z^{m-1} \leftarrow \cdots$$

has zero cohomology [14, Remark 2.7]. If $Z \gg 0$, then the cohomology groups of the bundles in the preceding complex vanish in positive degree, so that H^0 is an exact functor. Consequently the complex below also has zero cohomology, i.e., all of its Koszul groups vanish:

$$(9.2)\qquad H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1}) \leftarrow H^0(\Omega_Z^m)_0 \otimes H^0(\Omega_Z^m \otimes N_Z^{m-1}) \leftarrow \bigwedge^2 H^0(\Omega_Z^m)_0 \otimes H^0(N_Z^{m-1}) \leftarrow \cdots .$$

The problem is to establish vanishing for the first two cohomology groups of the quotient complex

$$(9.3)\qquad H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1})_J \leftarrow H^0(\Omega_Z^m)_0 \otimes H^0(\Omega_Z^m \otimes N_Z^{m-1})_J \leftarrow \bigwedge^2 H^0(\Omega_Z^m)_0 \otimes H^0(N_Z^{m-1})_J \leftarrow \cdots .$$

For the zeroth cohomology this is immediate, so that part (ii) is satisfied, as remarked above.

For the vanishing of H^1_{Koz} for the quotient sequence, suppose given an element $\overline{x}$ in the middle term of (9.2) which goes to zero. If the map

$$(9.4)\qquad H^0(\Omega_Z^m)_0 \otimes JH^0(\Omega_Z^m \otimes N_Z^{m-1}) \to JH^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1})$$

is surjective, then there is an x' in the coset $\overline{x}$ which goes to zero in (9.2), so that the vanishing of $H^1_{\mathrm{Koz}}(K_{\Omega^m}(N_Z)_J)$ follows from that of $H^1_{\mathrm{Koz}}(K_{\Omega^m}(N_Z))$. To prove the surjectivity of (9.4), consider the diagram below:

$$\begin{array}{ccc} H^0(\Omega_Z^m) \otimes H^0(\Omega_Z^m \otimes \Theta_Y \otimes N_Z^{m-2}) & \xrightarrow{\alpha} & H^0(\Omega_Z^m)_0 \otimes H^0(\Omega_Z^m \otimes N_Z^{m-1}) \\ \gamma\big\downarrow & & \big\downarrow\delta \\ H^0(\Omega_Z^{m\otimes 2} \otimes \Theta_Y \otimes N_Z^{m-2}) & \xrightarrow{\beta} & H^0(\Omega_Z^{m\otimes 2} \otimes N_Z^{m-1}) \end{array}$$

The assertion to be proved is equivalent to the relation $\mathrm{im}(\beta) = \mathrm{im}(\delta\alpha)$. To establish this latter fact it suffices to show that γ is surjective. We do this by lifting to Y:

$$\begin{array}{ccc} H^0(\Omega_Y^n(1)) \otimes H^0(\Omega_Y^n(m-1) \otimes \Theta_Y) & \longrightarrow & H^0(\Omega_Z^m)_0 \otimes H^0(\Omega_Z^m \otimes \Theta_Y \otimes N_Z^{m-2}) \\ \tilde{\gamma}\big\downarrow & & \big\downarrow\gamma \\ H^0(\Omega_Y^{n\otimes 2}(m) \otimes \Theta_Y) & \longrightarrow & H^0(\Omega_Z^{m\otimes 2} \otimes \Theta_Y \otimes N_Z^{m-2}) \end{array}$$

But $\tilde{\gamma}$, as well as the horizontal maps, are surjective for $Z \gg 0$, provided that $m > 2$, or, if $m = 2$, that $H^{3,0} = 0$, and so we are done.

10. Appendix: Weighted projective spaces

We apply the argument of [2] to derive divisibility and degree conditions which imply maximality for hypersurfaces in a weighted projective space [8].

The set $\mathscr{M}$ of possible pairs (degree, weight) $= (d, Q)$ so obtained is smaller than the set $\mathscr{L}\mathscr{T}$ of pairs for which local Torelli holds [19, Theorem 10.1], but it is nonetheless infinite: $\mathscr{M}$ contains infinitely many Q's and for each of these infinitely many d's.

We begin by recalling the standard notation. Let $Q = (q_0, \ldots, q_{n+1})$ be a vector of nonnegative integers, let $S = \mathbb{C}[X_0, \ldots, X_{n+1}]$ be the graded ring with generators X_i of degree q_i, and let $\mathbb{P}(Q) = \operatorname{Proj}(S)$ be the associated weighted projective space. Set

$$s = \sum q_i, \qquad m = \text{l.c.m.}(q_0, \ldots, q_{m+1}).$$

Fix a homogeneous element $F \in S$ of degree d, let J be the ideal generated by the partials of F, and let $R = S/J$. If F is quasi-smooth, i.e., if the subset of $\mathbb{C}^{n+2}$ defined by the vanishing of the partial derivatives consists of the origin alone, then R is a graded Artin ring with $R^j = 0$ for $j > \sigma = (n+2)d - 2s$, and with $R^\sigma \cong \mathbb{C}$. Let X be the hypersurface in $\mathbb{P}$ defined by F. By [8, 3.1.6], X is a V-manifold, and by [18] its cohomology carries a pure Hodge structure with

$$(*) \qquad H^{p,q} \cong R^{(q+1)d-s},$$

where $p + q = n$, with $n = \dim X$. Moreover, the space of infinitesimal deformations is identified with R^d, just as for ordinary projective hypersurfaces.

Verification of the conditions of Proposition (2.5) for hypersurfaces with $H^{n,0} \neq 0$ therefore reduce to the following:

(i) Nondegeneracy: surjectivity of the map $R^d \otimes R^{(q+1)d-s} \to R^{(q+2)d-s}$.

(ii) Local Torelli: Macaulay's theorem: injectivity of the map

$$R^d \to \operatorname{Hom}(R^{d-s}, R^{2d-s}).$$

(iii) Correct symmetrizer dimension: validity of the symmetrizer lemma [10] for the multiplication map $R^{d-s} \times R^{2d-s} \to R^{3d-2s}$.

Now let G be the Delorme number, defined by

$$G = -s + \frac{1}{n+1} \sum_{2 \le v \le n+2} \binom{n}{v-2} \sum_{|J|=v} \text{l.c.m.}(q_{j_1}, \ldots, q_{j_v}).$$

Then [6, Proposition 2.2, p. 207] asserts that

$$\text{if } l > G \quad \text{then } S^{km} \otimes S^l \to S^{km+l} \text{ is surjective.}$$

The relevant Macaulay Theorem and Symmetrizer Lemma also contain divisibility and degree conditions, the latter involving G, and are given in [19, Theorem 2.8, p. 40; 10, Theorem 1.6, p. 403].

Application of the above results require that *both* d and s be divisible by m. The set of Q for which these conditions hold is infinite: Let $Q' \in \mathbb{Z}^p$ be fixed, let $Q(b)'' \in \mathbb{Z}^q$ be the vector all of whose components are 1, and let $Q(b) = Q'Q(b)''$ be the concatenation. Let b_0 be the least nonnegative integer such that m divides $s(Q')+b$. Then all $Q(b)$ with b in the arithmetic progression $b_0 + km$ give weights of the required type.

The degree conditions corresponding to (i)–(iii) are

(i) $d - s > G$ or $d > G$,
(ii) $nd - s \geq \max(G+1, 0)$ or $d - s \geq G + 1$,
(iii) $(n-2)d > \max(G, -1) - 1$ or $(n-2)d > G - s - 1$.

In addition, we require $d \geq s$ so that $H^{n,0} \neq 0$. Thus, given Q satisfying the divisibility conditions, there is a computable integer $d_0(Q)$ such that $d \geq d_0$ implies that the variation defined by the pair (d, Q) is maximal. We remark [19, 2.6] that $G \leq -s + m(n+1)$.

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